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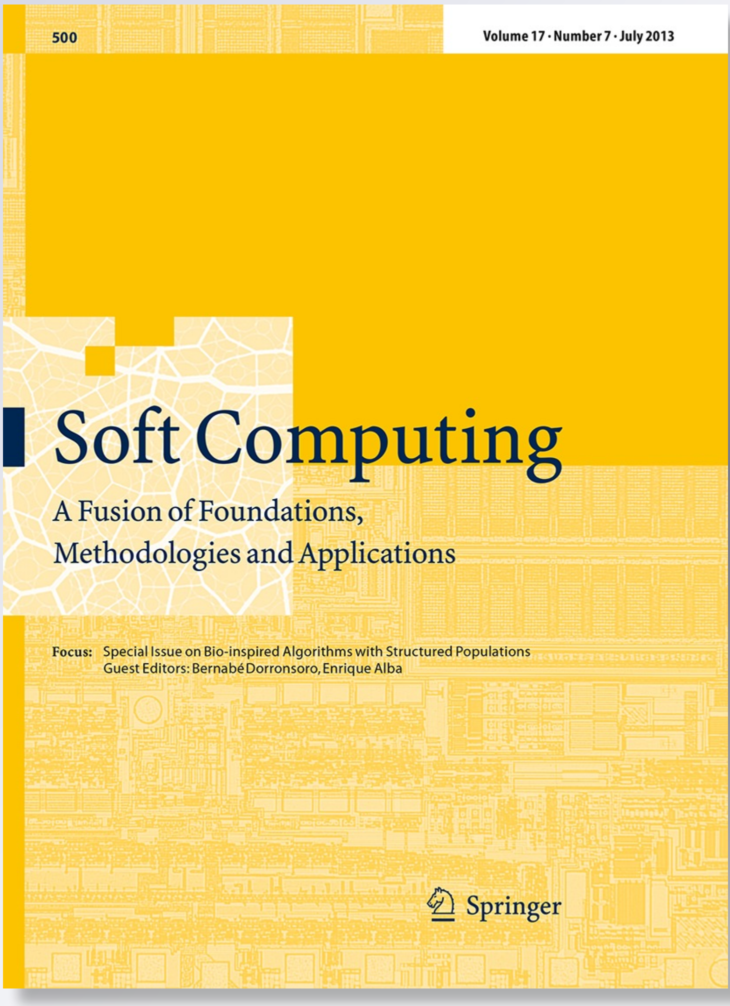
Soft Computing
A Fusion of Foundations, Methodologies and Applications

ISSN 1432-7643
Volume 17
Number 7

Soft Comput (2013) 17:1241-1252
DOI 10.1007/s00500-012-0979-1



Springer

Soft Comput (2013) 17:1241–1252
DOI 10.1007/s00500-012-0979-1

FOUNDATIONS

Multi-granulation rough sets based on tolerance relations

Weihua Xu · Qiaorong Wang · Xiantao Zhang

Published online: 24 January 2013

Abstract The original rough set model is primarily concerned with the approximations of sets described by a single equivalence relation on the universe. Some further investigations generalize the classical rough set model to rough set model based on a tolerance relation. From the granular computing point of view, the classical rough set theory is based on a single granulation. For some complicated issues, the classical rough set model was extended to multi-granulation rough set model (MGRS). This paper extends the single-granulation tolerance rough set model (SGTRS) to two types of multi-granulation tolerance rough set models (MGTRS). Some important properties of the two types of MGTRS are investigated. From the properties, it can be found that rough set model based on a single tolerance relation is a special instance of MGTRS. Moreover, the relationship and difference among SGTRS, the first type of MGTRS and the second type of MGTRS are discussed. Furthermore, several important measures are presented in two types of MGTRS, such as rough measure and quality of approximation. Several examples are considered to illustrate the two types of multi-granulation tolerance rough set models. The results from this research are both theoretically and practically meaningful for data reduction.

Keywords Rough set · Multi-granulation · Tolerance relation · Upper approximation · Lower approximation

1 Introduction

The classical rough set theory is based on the classification mechanism, from which the classification can be viewed as an equivalence relation. Knowledge granules induced by an equivalence relation in the classical rough set theory form a partition of the universe of discourse. In contrast, knowledge granules induced by an ordinary binary relation in generalized rough set theory form a covering of the universe of discourse. In rough set theory as well as generalized rough set theory, lower and upper approximations are constructed and any subset of universe of discourse can be expressed by them. Partition, covering, granulation and approximation are the methods widely used in human reasoning.

In practice, due to the existence of uncertainty and complexity, some particular problems cannot be settled perfectly by means of classical rough set. Therefore it is vital to generalize the classical rough set model. A key notion in Pawlak rough set model is equivalence relation. By replacing the equivalence relation with other binary relations, several extensions of rough set model have been proposed in terms of various requirements. The fuzzy rough set models and the rough fuzzy set models were proposed and studied by Dübois and Prade (1990) and Yao (2004) and later studied further by other researchers (Ouyang et al. 2010; Pei 2005). The rough set model based on similarity relation was illustrated in ref

Communicated by L. Spada.

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(Pomykala 2002; Slowinski and Vanderpooten 2000). Zakowski (1983) suggested that one can use a compatibility relation, also called tolerance relation, instead of an equivalence relation. The rough set model based on tolerance relation was also described in detail in the literature (Jarinen 2005; Kim 2001; Ouyang et al. 2010; Pomykala 1988, 2002; Skowron and Stepaniuk 1996; Xu et al. 2004; Yao and Lin 1996; Yao 2003; Zheng et al. 2005). Yao summarized some different generalized rough sets including models defined by arbitrary binary relations in the paper (Yao and Lin 1996; Yao 2003).

From the view of granular computing (Liang and Qian 2006; Ma et al. 2007; Qian et al. 2009; Yao 2000, 2005), a general concept described by a set is always characterized through upper and lower approximations under a single granulation, i.e., the concept is depicted by known knowledge induced from a single relation (like the classical equivalence relation, tolerance relation). The use of rough set model based on single granulation is limited to solving practical complex problems. Qian and Xu generalized them to multi-granulation rough set model which are based on multiply equivalence relations (Qian et al. 2010a, b, c; Xu et al. 2011a, b, 2012a, b) to adapt some practical applications that exist: contradiction and other problems like the following cases (Qian et al. 2010).

Case 1 In some data analysis issues, for the same object, there is a contradiction or inconsistent relationship between its values under one attributes set and those under another attributes set.

Case 2 In the process of some decision making, the decision or the view of each of decision makers may be independent for the same project.

Case 3 To extract decision rules from distributive information systems and groups of intelligent agents, for the reduction of the time complexity of rule extractions, it is unnecessary for us to perform the intersection operations in between all the sites in the context of distributive information systems.

In this paper, we will introduce another two types of multi-granulation rough set models which are based on multiple tolerance relations to solve more complicated problem. The main objective of this paper was to extend a rough set model based on a tolerance relation to a multi-granulation rough set model based on multiple tolerance relations. The rest of the paper is organized as follows: Some preliminary concepts about tolerance rough set such as the lower and upper approximations and accuracy measure are briefly reviewed in Sect. 2. In Sect. 3, we define two types of multi-granulation rough set models based on multiple tolerance relations, and some properties about such models are showed. With comparison to a tolerance multi-granulation rough set model, the tolerance single-granulation rough set model is a special instance. Moreover, we discuss the difference and relationship among single-granulation tolerance rough set (SGTRS), the first type of multi-granulation tolerance rough set (1st MGTRS) and the second type of multi-granulation tolerance rough set (2nd MGTRS) in Sect. 4. Furthermore, some measures are proposed in two types of MGTRS, such as rough measure, quality of approximation in Sect. 5. Finally, we conclude the paper briefly in Sect. 6.

2 Preliminaries

Let us first recall necessary concepts and preliminaries required in the sequel of our work. Detailed description of these theories can be found in the literature (Jarinen 2005; Yao and Lin 1996).

The notion of approximation space provides a convenient tool for the representation of objects in terms of their attribute values.

A tolerance approximation space is a tolerance relation system $K = (U, \mathcal{R})$, where U, called universe, is a finite non-empty set of objects, and $\mathcal{R}$ is a tolerance relation.

If a binary relation R on the universe U is reflexive and symmetric, it is called a tolerance relation on U. The set of all tolerance relations on U is denoted by Tol(U). Obviously, tolerance relation $R \in Tol(U)$ can construct a covering of the universe U denoted by $\hat{R}$, where a covering $\hat{R}$ on the universe U is a family of subsets of the universe which satisfied any subset of $\hat{R}$ is not empty and the unit of all subsets of $\hat{R}$ is U. For any tolerance relation $R \in Tol(U)$ and $x \in U$, the set

$$\hat{R}(x) = \{y \in U : xRy\}$$

is called the tolerance neighborhood of x.

Let U be a universe and $\hat{R}$ be a covering on U induced by a tolerance relation. The lower approximation and the upper approximation of a set $X \subseteq U$ are, respectively, defined by

$$X_{\hat{R}} = \{x \in U : \hat{R}(x) \subseteq X\},$$
$$X^{\hat{R}} = \{x \in U : \hat{R}(x) \cap X \neq \emptyset\}.$$

The set $\mathrm{Bn}_{\hat{R}}(x) = X^{\hat{R}} - X_{\hat{R}}$ is called the boundary of X.

The set $X_{\hat{R}}$ consists of elements which surely belong to X in view of the knowledge provided by R, while $X^{\hat{R}}$ consists of elements which possibly belong to X. The boundary is the actual area of uncertainty. It consists of elements whose membership in X cannot be decided when R-related objects can not be distinguished from each other.

Some basic properties of approximations are discussed in the following:

If $X_{\hat{R}} = X^{\hat{R}}$, the set $X \subseteq U$ is definable. Otherwise, the set X is indefinable. It's obvious that a set X is definable if and only if its boundary $\mathrm{Bn}_{\hat{R}}(X)$ is empty. The pair

$(X^{\hat{R}}, X_{\hat{R}})$ is called a tolerance rough set model. From the view of granule computing, we say the pair $(X^{\hat{R}}, X_{\hat{R}})$ is a single-granulation tolerance rough set model (in brief, SGTRS).

Let U be a universe and $\hat{R}$ is a covering on U induced by a tolerance relation R. If $X, Y \subseteq U$, X^c is the complement of X. Then the following properties hold:

1. $X_{\hat{R}} \subseteq X \subseteq X^{\hat{R}}$;
2. $\emptyset_{\hat{R}} = \emptyset^{\hat{R}} = \emptyset, U_{\hat{R}} = U^{\hat{R}} = U$;
3. $(X_{\hat{R}})^c = (X^c)^{\hat{R}}, (X^{\hat{R}})^c = (X^c)_{\hat{R}}$;
4. $\mathrm{Bn}_{\hat{R}}(X) = \mathrm{Bn}_{\hat{R}}(X^c)$.

To measure the imprecision of a rough set based on a tolerance relation, for $X \subseteq U$ and $X \neq \emptyset$, the following ratio:

$$\rho_{\hat{R}}(X) = 1 - \frac{|X_{\hat{R}}|}{|X^{\hat{R}}|},$$

is called the rough measure of X by tolerance relation R.

Let R and S be two tolerance relations, and $\hat{R}$ and $\hat{S}$ be the corresponding coverings on the universe U, respectively. If a class $\hat{R}(x)$ of $\hat{R}$ is a subset of a class $\hat{S}(x)$ of $\hat{S}$, the class $\hat{R}(x)$ is called deterministic with respect to $\hat{S}$, or just deterministic, if $\hat{S}$ is understood.

A frequently applied measure for this situation is the quality of approximation of $\hat{S}$ by $\hat{R}$, also called the degree of dependency. It is defined by

$$\gamma(\hat{R}, \hat{S}) = \frac{\Sigma\{|X_{\hat{R}}| : X \in \hat{S}\}}{|U|},$$

which evaluates the deterministic part of the rough set description of $\hat{S}$ by counting those elements that can be re-classified to blocks of $\hat{S}$ with the knowledge given by $\hat{R}$.

3 Two types of MGTRS

In this section, we will investigate two types of rough set models induced by several tolerance relations from the view of granule.

3.1 The first type of MGTRS

We first discuss the first type of two-granulation approximations of a concept by using two tolerance relations in an approximation space.

Definition 3.1 Let U be a universe and $\hat{R}, \hat{S}$ be two coverings of U induced by tolerance relations R and S. The first type of two-granulation tolerance lower approximation and the upper approximation of X on U are defined by the following:

$$FX_{\hat{R}+\hat{S}} = \{x \in U : \hat{R}(x) \subseteq X \text{ or } \hat{S}(x) \subseteq X\},$$
$$FX^{\hat{R}+\hat{S}} = \{x \in U : \hat{R}(x) \cap X \neq \emptyset \text{ and } \hat{S}(x) \cap X \neq \emptyset\}.$$

Moreover, if $FX_{\hat{R}+\hat{S}} \neq FX^{\hat{R}+\hat{S}}$, we say that X is the first type of tolerance rough set with respect to two granulations $\hat{R}$ and $\hat{S}$. Otherwise, we say that X is the first type of definable set with respect to two granulations $\hat{R}$ and $\hat{S}$.

The area of uncertainty or boundary region of this rough set is defined as

$$\mathrm{Bn}^F_{\hat{R}+\hat{S}}(X) = FX^{\hat{R}+\hat{S}} - FX_{\hat{R}+\hat{S}}.$$

It can be found that the first type of tolerance rough set with respect to two granulations will be SGTRS, if two granulations $\hat{R}$ and $\hat{S}$ satisfy $\hat{R} = \hat{S}$. That is to say, SGTRS is a particular case of the first type of tolerance two-granulation rough sets.

Example 3.1 Suppose that the set $U = \{1, 2, 3, 4, 5, 6\}$ consists of six cars called 1, 2, 3, 4, 5, 6, respectively. Their price, mileage, size and max-speed are given in Table 1.

Let us define one tolerance relation R_1 so that two cars are R_1-related if their price and size are, respectively, not completely different, another tolerance relation R_2 so that two cars are R_2-related if their size and max-speed are not completely different.

Let consider the lower and upper R-approximation of the set $X = \{3, 4, 5, 6\}$.

Obviously, we can obtain

$$\hat{R}_1(1) = \{1, 4, 5\},$$
$$\hat{R}_1(2) = \{2, 5, 6\},$$
$$\hat{R}_1(3) = \{3\},$$
$$\hat{R}_1(4) = \{1, 4, 5\},$$
$$\hat{R}_1(5) = \{1, 4, 5, 6\},$$
$$\hat{R}_1(6) = \{2, 5, 6\}.$$

That is to say $\hat{R}_1 = \{\{1, 4, 5\}, \{2, 5, 6\}, \{3\}, \{1, 4, 5\}, \{1, 4, 5, 6\}, \{2, 5, 6\}\}$. Then we have

Table 1 An information system about cars

Car	Price	Mileage	Size	Max-speed
1	{*high*}	{*low*}	{*full*}	{*low*}
2	{*low*}	{*high, low*}	{*full*}	{*low*}
3	{*high, low*}	{*high, low*}	{*compact*}	{*low*}
4	{*high*}	{*high, low*}	{*full*}	{*high*}
5	{*high, low*}	{*high, low*}	{*full*}	{*high*}
6	{*low*}	{*high*}	{*full*}	{*high, low*}

$X_{\hat{R}_1} = \{3\}$,
$X^{\hat{R}_1} = \{1,2,3,4,5,6\}$.

Additionally,

$\hat{R}_2(1) = \{1,2,6\}$,
$\hat{R}_2(2) = \{1,2,6\}$,
$\hat{R}_2(3) = \{3\}$,
$\hat{R}_2(4) = \{4,5,6\}$,
$\hat{R}_2(5) = \{4,5,6\}$,
$\hat{R}_2(6) = \{1,2,4,5,6\}$.

That is to say, $\hat{R}_2 = \{\{1,2,6\},\{1,2,6\},\{3\},\{4,5,6\},\{4,5,6\},\{1,2,4,5,6\}\}$. So the following results hold:
$X_{\hat{R}_2} = \{4,5\}$,
$X^{\hat{R}_2} = \{1,2,4,5,6\}$.

Furthermore, we have $\hat{R}_1 \cup \hat{R}_2 = \{\{1\},\{2,6\},\{3\},\{4,5\},\{4,5,6\},\{2,5,6\}\}$ from $\hat{R}_1$ and $\hat{R}_2$. Then we can obtain

$X_{\hat{R}_1 \cup \hat{R}_2} = \{2,4,5,6\}$,
$X^{\hat{R}_1 \cup \hat{R}_2} = \{2,4,5,6\}$.

From Definition 3.1, we can compute that the first type of two-granulation tolerance lower and upper approximations of X are

$FX_{\hat{R}_1+\hat{R}_2} = \{3,4,5\}$,
$FX^{\hat{R}_1+\hat{R}_2} = \{1,2,4,5,6\}$.

Obviously, the following can be found:

$X_{\hat{R}_1} \cup X_{\hat{R}_2} = FX_{\hat{R}_1+\hat{R}_2}$,
$X_{\hat{R}_1} \cap X_{\hat{R}_2} = FX^{\hat{R}_1+\hat{R}_2}$,
$FX_{\hat{R}_1+\hat{R}_2} \subseteq X_{\hat{R}_1 \cup \hat{R}_2} \subseteq X^{\hat{R}_1 \cup \hat{R}_2} \subseteq FX^{\hat{R}_1+\hat{R}_2}$.

In fact, we can obtain some properties of the first type of two-granulation tolerance rough sets in a tolerance approximation space.

Proposition 3.1 *Let U be a universe, $\hat{R}$ and $\hat{S}$ be two coverings of U induced by tolerance relations R and S and $X \subseteq U$. Then the following properties hold:*

1. $FX_{\hat{R}+\hat{S}} \subseteq X \subseteq FX^{\hat{R}+\hat{S}}$;
2. $(FX^{\hat{R}+\hat{S}})^c = (FX^c)_{\hat{R}+\hat{S}}, (FX_{\hat{R}+\hat{S}})^c = (FX^c)^{\hat{R}+\hat{S}}$;
3. $F\emptyset_{\hat{R}+\hat{S}} = F\emptyset^{\hat{R}+\hat{S}} = \emptyset, FU_{\hat{R}+\hat{S}} = FU^{\hat{R}+\hat{S}} = U$;
4. $\mathrm{Bn}^F_{\hat{R}+\hat{S}}(X) = \mathrm{Bn}^F_{\hat{R}+\hat{S}}(X^c)$.

Proof It is obvious that all terms hold when $\hat{R} = \hat{S}$. When $\hat{R} \neq \hat{S}$, the proposition can be proved as follows:

1a. For any $x \in FX_{\hat{R}+\hat{S}}$, it can be known that $\hat{R}(x) \subseteq X$ or $\hat{S}(x) \subseteq X$ by Definition 3.1. At the same time, $x \in \hat{R}(x)$ and $x \in \hat{S}(x)$ because of the reflexive of R and S. So we have $x \in X$. Hence, $FX_{\hat{R}+\hat{S}} \subseteq X$.

1b. For any $x \in X$, we have $x \in \hat{R}(x)$ and $x \in \hat{S}(x)$. So we have $\hat{R}(x) \cap X \neq \emptyset$ and $\hat{S}(x) \cap X \neq \emptyset$, that is to say $x \in FX^{\hat{R}+\hat{S}}$. Hence, $X \subseteq FX^{\hat{R}+\hat{S}}$ holds. From (1a) and (1b), $FX_{\hat{R}+\hat{S}} \subseteq X \subseteq FX^{\hat{R}+\hat{S}}$.

2a. For any $x \in (FX^c)_{\hat{R}+\hat{S}}$, we have

$$\begin{aligned} x \in (FX^c)_{\hat{R}+\hat{S}} &\Longleftrightarrow \hat{R}(x) \subseteq X^c \text{ or } \hat{S}(x) \subseteq X^c \\ &\Longleftrightarrow \hat{R}(x) \cap X = \emptyset \text{ or } \hat{S}(x) \cap X = \emptyset \\ &\Longleftrightarrow x \notin FX^{\hat{R}+\hat{S}} \\ &\Longleftrightarrow x \in (FX^{\hat{R}+\hat{S}})^c, \end{aligned}$$

Hence, $(FX^{\hat{R}+\hat{S}})^c = (FX^c)_{\hat{R}+\hat{S}}$.

2b. From $(FX^{\hat{R}+\hat{S}})^c = (FX^c)_{\hat{R}+\hat{S}}$, we can have that $((FX^c)^{\hat{R}+\hat{S}})^c = ((FX^c)^c)_{\hat{R}+\hat{S}}$, i.e., $(FX_{\hat{R}+\hat{S}})^c = (FX^c)^{\hat{R}+\hat{S}}$.

3a. From $FX_{\hat{R}+\hat{S}} \subseteq X$, we have $F\emptyset_{\hat{R}+\hat{S}} \subseteq \emptyset$. Besides, it is well known that $\emptyset \subseteq F\emptyset_{\hat{R}+\hat{S}}$, so $F\emptyset_{\hat{R}+\hat{S}} = \emptyset$. If $F\emptyset^{\hat{R}+\hat{S}} \neq \emptyset$, there must exist $y \in F\emptyset^{\hat{R}+\hat{S}}$ such that $\hat{R}(y) \cap \emptyset \neq \emptyset$ and $\hat{S}(x) \cap X \neq \emptyset$. Apparently, this is a contradiction. Therefore, $F\emptyset^{\hat{R}+\hat{S}} = \emptyset$. From above, $F\emptyset_{\hat{R}+\hat{S}} = F\emptyset^{\hat{R}+\hat{S}} = \emptyset$ holds.

3b. From the duality of $FX_{\hat{R}+\hat{S}}$ and $FX^{\hat{R}+\hat{S}}$, $(FU_{\hat{R}+\hat{S}})^c = (FU^c)^{\hat{R}+\hat{S}} = F\emptyset^{\hat{R}+\hat{S}} = \emptyset$. Hence, $FU_{\hat{R}+\hat{S}} = U$. Similarly, $FU^{\hat{R}+\hat{S}} = U$.

4. From the definition of boundary region of the rough set and the duality of $FX_{\hat{R}+\hat{S}}$ and $FX^{\hat{R}+\hat{S}}$, we know that

$$\begin{aligned} \mathrm{Bn}^F_{\hat{R}+\hat{S}}(X^c) &= (FX^c)^{\hat{R}+\hat{S}} - (FX^c)_{\hat{R}+\hat{S}} \\ &= (FX_{\hat{R}+\hat{S}})^c - (FX^{\hat{R}+\hat{S}})^c \\ &= (U - FX_{\hat{R}+\hat{S}}) - (U - FX^{\hat{R}+\hat{S}}) \\ &= FX^{\hat{R}+\hat{S}} - FX_{\hat{R}+\hat{S}} \\ &= \mathrm{Bn}^F_{\hat{R}+\hat{S}}(X). \end{aligned}$$

So, $\mathrm{Bn}^F_{\hat{R}+\hat{S}}(X) = \mathrm{Bn}^F_{\hat{R}+\hat{S}}(X^c)$. □

Thus, the proposition was proved. □

Proposition 3.1 shows the first two-granulation tolerance rough set satisfies the basic properties as rough set in tolerance approximation space. For example, (1) embodies that the 1st two-granulation tolerance lower approximations are included into the target concept, and the upper approximations include the target concept; (2) shows the duality between the lower and upper approximations;

(3) expresses the normality and conormality of the first two-granulation tolerance rough sets; (4) reflects that the uncertainty of the concept X and its complement is totaly the same.

To discover the relationship between the first type of two-granulation approximation based on tolerance relation of a single set and that of two sets described on the universe U, the following properties are given:

Proposition 3.2 *Let U be a universe, $\hat{R}$ and $\hat{S}$ be two coverings of U induced by tolerance relations R and S and $X, Y \subseteq U$. Then the following properties hold:*

1. $F(X \cap Y)_{\hat{R}+\hat{S}} \subseteq FX_{\hat{R}+\hat{S}} \cap FY_{\hat{R}+\hat{S}}$;
2. $F(X \cup Y)^{\hat{R}+\hat{S}} \supseteq FX^{\hat{R}+\hat{S}} \cup FY^{\hat{R}+\hat{S}}$;
3. $X \subseteq Y \Longrightarrow FX^{\hat{R}+\hat{S}} \subseteq FY^{\hat{R}+\hat{S}}$;
4. $X \subseteq Y \Longrightarrow FX_{\hat{R}+\hat{S}} \subseteq FY_{\hat{R}+\hat{S}}$;
5. $F(X \cap Y)^{\hat{R}+\hat{S}} \subseteq FX^{\hat{R}+\hat{S}} \cap FY^{\hat{R}+\hat{S}}$;
6. $F(X \cup Y)_{\hat{R}+\hat{S}} \supseteq FX_{\hat{R}+\hat{S}} \cup FY_{\hat{R}+\hat{S}}$.

Proof It is obvious that all terms hold when $\hat{R} = \hat{S}$ or $X = Y$. When $\hat{R} \neq \hat{S}$ and $X \neq Y$, the proposition can be proved as follows:

1. For any $x \in F(X \cap Y)_{\hat{R}+\hat{S}}$, we have that $\hat{R}(x) \subseteq X \cap Y$ or $\hat{S}(x) \subseteq X \cap Y$ by Definition 3.1. Then, one can find that $\hat{R}(x) \subseteq X$ and $\hat{R}(x) \subseteq Y$ hold at the same time or $\hat{S}(x) \subseteq X$ and $\hat{S}(x) \subseteq Y$ hold at the same time. Then $\hat{R}(x) \subseteq X$ or $\hat{S}(x) \subseteq X$ holds, and $\hat{R}(x) \subseteq Y$ or $\hat{S}(x) \subseteq Y$ holds. That is to say $x \in FX_{\hat{R}+\hat{S}}$ and $x \in FY_{\hat{R}+\hat{S}}$, i.e., $x \in FX_{\hat{R}+\hat{S}} \cap FY_{\hat{R}+\hat{S}}$. Hence, $F(X \cap Y)_{\hat{R}+\hat{S}} \subseteq FX_{\hat{R}+\hat{S}} \cap FY_{\hat{R}+\hat{S}}$.
2. For any $x \in FX^{\hat{R}+\hat{S}} \cup FY^{\hat{R}+\hat{S}}$, $x \in FX^{\hat{R}+\hat{S}}$ or $x \in FY^{\hat{R}+\hat{S}}$. Then, $\hat{R}(x) \cap X \neq \emptyset$ and $\hat{S}(x) \cap X \neq \emptyset$ hold, or $\hat{R}(x) \cap Y \neq \emptyset$ and $\hat{S}(x) \cap Y \neq \emptyset$ hold. So not only $\hat{R}(x) \cap (X \cap Y) \neq \emptyset$ holds, but $\hat{S}(x) \cap (X \cap Y) \neq \emptyset$. That is to say, $x \in F(X \cup Y)^{\hat{R}+\hat{S}}$. Hence, $F(X \cup Y)^{\hat{R}+\hat{S}} \supseteq FX^{\hat{R}+\hat{S}} \cup FY^{\hat{R}+\hat{S}}$.
3. For any $x \in FX^{\hat{R}+\hat{S}}, \hat{R}(x) \cap X \neq \emptyset$ and $\hat{S}(x) \cap X \neq \emptyset$. When $X \subseteq Y$ holds, $\hat{R}(x) \cap Y \neq \emptyset$ and $\hat{S}(x) \cap Y \neq \emptyset$. Hence, $x \in FY^{\hat{R}+\hat{S}}$. Then we have $FX^{\hat{R}+\hat{S}} \subseteq FY^{\hat{R}+\hat{S}}$.
4. The proof can be obtained similarly to (3) by Definition 3.1.
5. Since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, by (3) it can be obtained that

   $$F(X \cap Y)^{\hat{R}+\hat{S}} \subseteq FX^{\hat{R}+\hat{S}},$$
   $$F(X \cap Y)^{\hat{R}+\hat{S}} \subseteq FY^{\hat{R}+\hat{S}}.$$

   Hence, $F(X \cap Y)^{\hat{R}+\hat{S}} \subseteq FX^{\hat{R}+\hat{S}} \cap FY^{\hat{R}+\hat{S}}$.
6. Since $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$, by (4) it can be obtained that

   $$FX_{\hat{R}+\hat{S}} \subseteq F(X \cup Y)_{\hat{R}+\hat{S}},$$
   $$FY_{\hat{R}+\hat{S}} \subseteq F(X \cup Y)_{\hat{R}+\hat{S}}.$$

   Hence, $F(X \cup Y)_{\hat{R}+\hat{S}} \supseteq FX_{\hat{R}+\hat{S}} \cup FY_{\hat{R}+\hat{S}}$.

The proof of the proposition is completed. □

This proposition reflects the properties about the approximations of two different concepts. Especially, the first and second items explain the two-granulation tolerance lower approximation is included in the intersection of the two single-granulation lower approximations while the upper one includes the union of the two single-granulation upper approximations, which are different from the multiplicativity and additivity of the single-granulation rough set.

Based on the above conclusions, we here extend SGTRS to the first multi-granulation tolerance rough set, where the set approximations are defined through using multiple tolerance relations on the universe.

Definition 3.2 Let U be a universe and $\hat{R}_i (i = 1, ..., m)$ be m coverings of U induced by tolerance relations R_i ($i = 1, ..., m$). The first type of multi-granulation tolerance lower and the upper approximations of X on U are defined by the following:

$$FX_{\sum_{i=1}^{m} \hat{R}_i} = \{x \in U : \vee_{i=1}^{m} \hat{R}_i(x) \subseteq X\},$$
$$FX^{\sum_{i=1}^{m} \hat{R}_i} = \{x \in U : \wedge_{i=1}^{m} \hat{R}_i(x) \cap X \neq \emptyset\},$$

where "$\vee$" means "some" and "$\wedge$" means "all".

Moreover, if $FX_{\sum_{i=1}^{m} \hat{R}_i} \neq FX^{\sum_{i=1}^{m} \hat{R}_i}$, we say that X is the first type of tolerance rough set with respect to multiple granulations R_i ($i = 1, ..., m$). Otherwise, we say X is the first type of tolerance definable set with respect to these multiple granulations.

Similarly, the area of uncertainty or boundary region of the first type of multi-granulation tolerance rough set is defined as

$$\mathrm{Bn}^{F}_{\sum_{i=1}^{m} \hat{R}_i}(X) = FX^{\sum_{i=1}^{m} \hat{R}_i} - FX_{\sum_{i=1}^{m} \hat{R}_i}.$$

To describe conveniently in our context, we express the first type of multi-granulation tolerance rough set using the first MGTRS. Moreover, one can obtain the following properties of the 1st MGTRS approximations:

Proposition 3.3 *Let U be a universe, $\hat{R}_i (i = 1, ..., m)$ be m coverings of U induced by tolerance relations R_i ($i = 1, ..., m$) and $X \subseteq U$. Then the following properties hold:*

1. $FX_{\sum_{i=1}^{m}\hat{R}_i} \subseteq X \subseteq FX^{\sum_{i=1}^{m}\hat{R}_i}$;
2. $F\emptyset_{\sum_{i=1}^{m}\hat{R}_i} = F\emptyset^{\sum_{i=1}^{m}\hat{R}_i} = \emptyset, FU_{\sum_{i=1}^{m}\hat{R}_i} = FU^{\sum_{i=1}^{m}\hat{R}_i} = U$;
3. $(FX_{\sum_{i=1}^{m}\hat{R}_i})^c = (FX^c)^{\sum_{i=1}^{m}\hat{R}_i}, (FX^{\sum_{i=1}^{m}\hat{R}_i})^c = (FX^c)_{\sum_{i=1}^{m}\hat{R}_i}$;
4. $\mathrm{Bn}^F_{\sum_{i=1}^{m}\hat{R}_i}(X) = \mathrm{Bn}^F_{\sum_{i=1}^{m}\hat{R}_i}(X^c)$.

Proof The proofs of these terms are similar to Proposition 3.1. □

Proposition 3.4 *Let U be a universe, $\hat{R}_i (i = 1, \ldots, m)$ be m coverings of U induced by tolerance relations R_i $(i = 1, \ldots, m)$ and $X, Y \subseteq U$. Then the following properties hold:*

1. $F(X \cap Y)_{\sum_{i=1}^{m}\hat{R}_i} \subseteq FX_{\sum_{i=1}^{m}\hat{R}_i} \cap FY_{\sum_{i=1}^{m}\hat{R}_i}$;
2. $F(X \cup Y)^{\sum_{i=1}^{m}\hat{R}_i} \supseteq FX^{\sum_{i=1}^{m}\hat{R}_i} \cup FY^{\sum_{i=1}^{m}\hat{R}_i}$;
3. $X \subseteq Y \Longrightarrow FX^{\sum_{i=1}^{m}\hat{R}_i} \subseteq FY^{\sum_{i=1}^{m}\hat{R}_i}$;
4. $X \subseteq Y \Longrightarrow FX_{\sum_{i=1}^{m}\hat{R}_i} \subseteq FY_{\sum_{i=1}^{m}\hat{R}_i}$;
5. $F(X \cap Y)^{\sum_{i=1}^{m}\hat{R}_i} \subseteq FX^{\sum_{i=1}^{m}\hat{R}_i} \cap FY^{\sum_{i=1}^{m}\hat{R}_i}$;
6. $F(X \cup Y)_{\sum_{i=1}^{m}\hat{R}_i} \supseteq FX_{\sum_{i=1}^{m}\hat{R}_i} \cup FY_{\sum_{i=1}^{m}\hat{R}_i}$.

Proof The proofs of these items are similar to Proposition 3.2. □

3.2 The second type of MGTRS

In this subsection, we will propose another type of MGTRS. We first introduce the second type of two-granulation approximations of a target set.

Definition 3.3 Let U be a universe and $\hat{R}$, $\hat{S}$ be two coverings of U induced by tolerance relations R and S. The second type of two-granulation lower approximations and the upper approximations of X on U are defined by the following:

$$SX_{\hat{R}+\hat{S}} = \{x \in U : \hat{R}(x) \subseteq X \text{ and } \hat{S}(x) \subseteq X\},$$
$$SX^{\hat{R}+\hat{S}} = \{x \in U : \hat{R}(x) \cap X \neq \emptyset \text{ or } \hat{S}(x) \cap X \neq \emptyset\}.$$

Moreover, if $SX_{\hat{R}+\hat{S}} \neq SX^{\hat{R}+\hat{S}}$, we say that X is the second type of rough (or indefinable) set with respect to two granulations $\hat{R}$ and $\hat{S}$. Otherwise, we say that X is the second type of definable set with respect to two granulations $\hat{R}$ and $\hat{S}$.

The area of uncertainty or boundary region of this rough set is defined as

$$\mathrm{Bn}^S_{\hat{R}+\hat{S}}(X) = SX^{\hat{R}+\hat{S}} - SX_{\hat{R}+\hat{S}}.$$

Example 3.2 (*Continued from Example 3.1*) From Example 3.1, we know that

$$\hat{R}_1 = \{\{1,4,5\},\{2,5,6\},\{3\},\{1,4,5\},\{1,4,5,6\},\{2,5,6\}\},$$
$$\hat{R}_2 = \{\{1,2,6\},\{1,2,6\},\{3\},\{4,5,6\},\{4,5,6\},\{1,2,4,5,6\}\},$$
$$\hat{R}_1 \cup \hat{R}_2 = \{\{1\},\{2,6\},\{3\},\{4,5\},\{4,5,6\},\{2,5,6\}\}.$$

And if takes $X = \{3, 4, 5, 6\}$ again, then by computing we can obtain that

$$SX_{\hat{R}_1+\hat{R}_2} = \emptyset$$

and

$$SX^{\hat{R}_1+\hat{R}_2} = \{1,2,3,4,5,6\}$$

are the second type of two-granulation lower and upper approximations of X, respectively. From the results of Example 3.1,

$$X_{\hat{R}_1} = \{3\},$$
$$X^{\hat{R}_1} = \{1,2,3,4,5,6\};$$
$$X_{\hat{R}_2} = \{4,5\},$$
$$X^{\hat{R}_2} = \{1,2,4,5,6\};$$
$$X_{\hat{R}_1\cup\hat{R}_2} = \{2,4,5,6\},$$
$$X^{\hat{R}_1\cup\hat{R}_2} = \{2,4,5,6\};$$

it is easily to find that

$$X_{\hat{R}_1} \cap X_{\hat{R}_2} = SX_{\hat{R}_1+\hat{R}_2},$$
$$X_{\hat{R}_1} \cup X_{\hat{R}_2} = SX^{\hat{R}_1+\hat{R}_2},$$
$$SX_{\hat{R}_1+\hat{R}_2} \subseteq X_{\hat{R}_1\cup\hat{R}_2} \subseteq X^{\hat{R}_1\cup\hat{R}_2} \subseteq SX^{\hat{R}_1+\hat{R}_2}.$$

Moreover, from Definition 3.3, we can obtain some properties in the second type of two-granulation tolerance rough sets in an approximation space.

Proposition 3.5 *Let U be a universe, $\hat{R}$ and $\hat{S}$ be two coverings of U induced by tolerance relations R and S and $X \subseteq U$. Then the following properties hold:*

1. $SX_{\hat{R}+\hat{S}} \subseteq X \subseteq SX^{\hat{R}+\hat{S}}$;
2. $(SX^{\hat{R}+\hat{S}})^c = (SX^c)_{\hat{R}+\hat{S}}, (SX_{\hat{R}+\hat{S}})^c = (SX^c)^{\hat{R}+\hat{S}}$;
3. $S\emptyset_{\hat{R}+\hat{S}} = S\emptyset^{\hat{R}+\hat{S}} = \emptyset, SU_{\hat{R}+\hat{S}} = SU^{\hat{R}+\hat{S}} = U$;
4. $\mathrm{Bn}^S_{\hat{R}+\hat{S}}(X) = \mathrm{Bn}^S_{\hat{R}+\hat{S}}(X^c)$.

Proof It is obvious that all terms hold when $\hat{R} = \hat{S}$. When $\hat{R} \neq \hat{S}$, the proposition can be proved as follows:

1a. For any $x \in SX_{\hat{R}+\hat{S}}$, it can be known that $\hat{R}(x) \subseteq X$ and $\hat{S}(x) \subseteq X$ by Definition 3.3. At the same time,

$x \in \hat{R}(x)$ and $x \in \hat{S}(x)$ because of the reflexive of R and S. So we have that $x \in X$. Hence, $SX_{\hat{R}+\hat{S}} \subseteq X$.

1b. For any $x \in X$, we have $x \in \hat{R}(x)$ and $x \in \hat{S}(x)$. So we have $\hat{R}(x) \cap X \neq \emptyset$ and $\hat{S}(x) \cap X \neq \emptyset$, that is to say $x \in SX^{\hat{R}+\hat{S}}$. Hence, $X \subseteq SX^{\hat{R}+\hat{S}}$ holds. From (1a) and (1b), $SX_{\hat{R}+\hat{S}} \subseteq X \subseteq SX^{\hat{R}+\hat{S}}$ is true.

2a. For any $x \in (SX^c)_{\hat{R}+\hat{S}}$, we have

$$\begin{aligned} x \in (SX^c)_{\hat{R}+\hat{S}} &\iff \hat{R}(x) \subseteq X^c \text{ and } \hat{S}(x) \subseteq X^c \\ &\iff \hat{R}(x) \cap X = \emptyset \text{ and } \hat{S}(x) \cap X = \emptyset \\ &\iff x \notin SX^{\hat{R}+\hat{S}} \\ &\iff x \in (SX^{\hat{R}+\hat{S}})^c, \end{aligned}$$

Hence, $(SX_{\hat{R}+\hat{S}})^c = (SX^c)^{\hat{R}+\hat{S}}$.

2b. From $(SX^{\hat{R}+\hat{S}})^c = (SX^c)_{\hat{R}+\hat{S}}$, we can have that $((SX^c)^{\hat{R}+\hat{S}})^c = ((SX^c)^c)_{\hat{R}+\hat{S}}$, i.e., $(SX^{\hat{R}+\hat{S}})^c = (SX^c)_{\hat{R}+\hat{S}}$.

3a. From $SX_{\hat{R}+\hat{S}} \subseteq X$, we have $S\emptyset_{\hat{R}+\hat{S}} \subseteq \emptyset$. Besides, it is well known that $\emptyset \subseteq S\emptyset_{\hat{R}+\hat{S}}$, so $S\emptyset_{\hat{R}+\hat{S}} = \emptyset$. If $F\emptyset^{\hat{R}+\hat{S}} \neq \emptyset$, there must exist $y \in F\emptyset^{\hat{R}+\hat{S}}$ such that $\hat{R}(y) \cap \emptyset \neq \emptyset$ and $\hat{S}(x) \cap X \neq \emptyset$. Apparently, this is a contradiction. Therefore, $F\emptyset^{\hat{R}+\hat{S}} = \emptyset$. From the above, $F\emptyset_{\hat{R}+\hat{S}} = F\emptyset^{\hat{R}+\hat{S}} = \emptyset$ holds.

3b. From the duality of $SX_{\hat{R}+\hat{S}}$ and $SX^{\hat{R}+\hat{S}}$, $(SU_{\hat{R}+\hat{S}})^c = (SU^c)^{\hat{R}+\hat{S}} = S\emptyset^{\hat{R}+\hat{S}} = \emptyset$. Hence, $SU_{\hat{R}+\hat{S}} = U$. Similarly, $SU^{\hat{R}+\hat{S}} = U$.

4. From the definition of boundary region of the rough set and the duality of $SX_{\hat{R}+\hat{S}}$ and $SX^{\hat{R}+\hat{S}}$, we know that

$$\begin{aligned} \mathrm{Bn}^S_{\hat{R}+\hat{S}}(X^c) &= (SX^c)^{\hat{R}+\hat{S}} - (SX^c)_{\hat{R}+\hat{S}} \\ &= (SX_{\hat{R}+\hat{S}})^c - (SX^{\hat{R}+\hat{S}})^c \\ &= (U - SX_{\hat{R}+\hat{S}}) - (U - SX^{\hat{R}+\hat{S}}) \\ &= SX^{\hat{R}+\hat{S}} - SX_{\hat{R}+\hat{S}} \\ &= \mathrm{Bn}^S_{\hat{R}+\hat{S}}(X). \end{aligned}$$

So, $\mathrm{Bn}^S_{\hat{R}+\hat{S}}(X) = \mathrm{Bn}^S_{\hat{R}+\hat{S}}(X^c)$. □

Thus, the proposition was proved.

Proposition 3.5 shows the second two-granulation tolerance rough set also satisfies the basic properties as tolerance rough set. For example, (1) embodies that the second two-granulation tolerance lower approximations satisfy the contraction and extension, respectively; (2) shows the duality between the lower and upper approximations; (3) expresses the normality and conormality of the first two-granulation tolerance rough sets; (4) reflects that the uncertainty of the concept X and its complement is totaly the same.

To discover the relationship between the second MGTRS of a single set and that of two sets described on the universe U, The following properties are given:

Proposition 3.6 *Let U be a universe, $\hat{R}$ and $\hat{S}$ be two coverings of U induced by tolerance relations R and S and $X, Y \subseteq U$. Then the following properties hold:*

1. $S(X \cap Y)_{\hat{R}+\hat{S}} = SX_{\hat{R}+\hat{S}} \cap SY_{\hat{R}+\hat{S}}$;
2. $S(X \cup Y)^{\hat{R}+\hat{S}} = SX^{\hat{R}+\hat{S}} \cup SY^{\hat{R}+\hat{S}}$;
3. $X \subseteq Y \Longrightarrow SX^{\hat{R}+\hat{S}} \subseteq SY^{\hat{R}+\hat{S}}$;
4. $X \subseteq Y \Longrightarrow SX_{\hat{R}+\hat{S}} \subseteq SY_{\hat{R}+\hat{S}}$;
5. $S(X \cap Y)^{\hat{R}+\hat{S}} \subseteq SX^{\hat{R}+\hat{S}} \cap SY^{\hat{R}+\hat{S}}$;
6. $S(X \cup Y)_{\hat{R}+\hat{S}} \supseteq SX_{\hat{R}+\hat{S}} \cup SY_{\hat{R}+\hat{S}}$.

Proof It is obvious that all terms hold when $\hat{R} = \hat{S}$ or $X = Y$. When $\hat{R} \neq \hat{S}$ and $X \neq Y$, the proposition can be proved as follows:

1. For any $x \in S(X \cap Y)_{\hat{R}+\hat{S}}$, by Definition 3.3 we have that

$$\begin{aligned} x \in S(X \cap Y)_{\hat{R}+\hat{S}} &\iff \hat{R}(x) \subseteq X \cap Y \text{ and } \hat{S}(x) \subseteq X \cap Y \\ &\iff \hat{R}(x) \subseteq X, \hat{R}(x) \subseteq Y, \hat{S}(x) \subseteq X \text{ and } \hat{S}(x) \subseteq Y \\ &\iff \hat{R}(x) \subseteq X, \hat{S}(x) \subseteq X, \hat{R}(x) \subseteq Y \text{ and } \hat{S}(x) \subseteq Y \\ &\iff x \in SX_{\hat{R}+\hat{S}} \text{ and } x \in SY_{\hat{R}+\hat{S}} \\ &\iff x \in SX_{\hat{R}+\hat{S}} \cap SY_{\hat{R}+\hat{S}} \end{aligned}$$

Hence, $S(X \cap Y)_{\hat{R}+\hat{S}} = SX_{\hat{R}+\hat{S}} \cap SY_{\hat{R}+\hat{S}}$.

2. For any $x \in S(X \cup Y)_{\hat{R}+\hat{S}}$, by Definition 3.3 we have that

$$\begin{aligned} x \in S(X \cup Y)^{\hat{R}+\hat{S}} &\iff \hat{R}(x) \cap (X \cap Y) \neq \emptyset \text{ or } \hat{S}(x) \cap (X \cap Y) \neq \emptyset \\ &\iff \hat{R}(x) \cap X \neq \emptyset \text{ or } \hat{R}(x) \cap Y \neq \emptyset \\ &\qquad \text{or } \hat{S}(x) \cap X \neq \emptyset \text{ or } \hat{S}(x) \cap Y \neq \emptyset \\ &\iff \hat{R}(x) \cap X \neq \emptyset \text{ or } \hat{S}(x) \cap X \neq \emptyset \\ &\qquad \text{or } \hat{R}(x) \cap Y \neq \emptyset \text{ or } \hat{S}(x) \cap Y \neq \emptyset \\ &\iff x \in SX^{\hat{R}+\hat{S}} \text{ or } x \in SY^{\hat{R}+\hat{S}} \\ &\iff x \in SX^{\hat{R}+\hat{S}} \cup SY^{\hat{R}+\hat{S}} \end{aligned}$$

Hence, $S(X \cup Y)^{\hat{R}+\hat{S}} = SX^{\hat{R}+\hat{S}} \cup SY^{\hat{R}+\hat{S}}$.

3. For any $x \in SX^{\hat{R}+\hat{S}}, \hat{R}(x) \cap X \neq \emptyset$ or $\hat{S}(x) \cap X \neq \emptyset$. When $X \subseteq Y$ holds, $\hat{R}(x) \cap Y \neq \emptyset$ or $\hat{S}(x) \cap Y \neq \emptyset$. Hence, $x \in SY^{\hat{R}+\hat{S}}$. Then we have $SX^{\hat{R}+\hat{S}} \subseteq SY^{\hat{R}+\hat{S}}$.
4. The proof can be obtained similarly to (3) by Definition 3.3.
5. Since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, by (3) it can be obtained that

$$S(X \cap Y)^{\hat{R}+\hat{S}} \subseteq SX^{\hat{R}+\hat{S}},$$
$$S(X \cap Y)^{\hat{R}+\hat{S}} \subseteq SY^{\hat{R}+\hat{S}}.$$

Hence, $S(X \cap Y)^{\hat{R}+\hat{S}} \subseteq SX^{\hat{R}+\hat{S}} \cap SY^{\hat{R}+\hat{S}}$.

6. Since $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$, by (4) it can be obtained that

$$SX_{\hat{R}+\hat{S}} \subseteq S(X \cup Y)_{\hat{R}+\hat{S}},$$
$$SY_{\hat{R}+\hat{S}} \subseteq S(X \cup Y)_{\hat{R}+\hat{S}}.$$

Hence, $S(X \cup Y)_{\hat{R}+\hat{S}} \supseteq SX_{\hat{R}+\hat{S}} \cup SY_{\hat{R}+\hat{S}}$.

Thus, the proposition is proved. □

This proposition reflects the properties about the approximations of two different concepts. The first and second items explain the two-granulation tolerance lower approximation equal to the intersection of the two single-granulation lower approximations and the upper one is consistent with the union of the two single-granulation upper approximations, which are different from the multiplicativity and additivity of the first two-granulation tolerance rough set.

Based on the above conclusions, we here extend the SGTRS to the second type of multi-granulation tolerance rough set, where the set approximations are defined through multiple tolerance relations on the universe.

Definition 3.4 Let U be a universe and $\hat{R}_i (i = 1, ..., m)$ be m coverings of U induced by tolerance relations R_i $(i = 1, ..., m)$. The second type of multi-granulation lower and the upper approximations of X on U are defined by the following

$$SX_{\sum_{i=1}^{m} \hat{R}_i} = \{x \in U : \wedge_{i=1}^{m} \hat{R}_i(x) \subseteq X\},$$
$$SX^{\sum_{i=1}^{m} \hat{R}_i} = \{x \in U : \vee_{i=1}^{m} \hat{R}_i(x) \cap X \neq \emptyset\},$$

where "$\vee$" means "some" and "$\wedge$" means "all".

Moreover, if $SX_{\sum_{i=1}^{m} \hat{R}_i} \neq SX^{\sum_{i=1}^{m} \hat{R}_i}$, we say that X is the second type of tolerance rough set with respect to multiple granulations R_i $(i = 1, ..., m)$. Otherwise, we say X is the second type of tolerance definable set with respect to these multiple granulations.

Similarly, the area of uncertainty or boundary region of the second type of multi-granulation rough set is defined as

$$\mathrm{Bn}^{S}_{\sum_{i=1}^{m} \hat{R}_i}(X) = SX^{\sum_{i=1}^{m} \hat{R}_i} - SX_{\sum_{i=1}^{m} \hat{R}_i}.$$

In order to describe conveniently in our context, we express the second type of multi-granulation tolerance rough set using the second MGTRS. Moreover, one can obtain the following properties of the 2nd MGTRS approximations:

Proposition 3.7 *Let U be a universe and $\hat{R}_i (i = 1, ..., m)$ be m coverings of U induced by tolerance relations R_i $(i = 1, ..., m)$ and $X \subseteq U$. Then the following properties hold:*

1. $SX_{\sum_{i=1}^{m} \hat{R}_i} \subseteq X \subseteq SX^{\sum_{i=1}^{m} \hat{R}_i}$;
2. $S\emptyset_{\sum_{i=1}^{m} \hat{R}_i} = S\emptyset^{\sum_{i=1}^{m}} \hat{R}_i = \emptyset, SU_{\sum_{i=1}^{m} \hat{R}_i} = SU^{\sum_{i=1}^{m} \hat{R}_i} = U$;
3. $\left(SX_{\sum_{i=1}^{m} \hat{R}_i}\right)^c = (SX^c)^{\sum_{i=1}^{m} \hat{R}_i}$, $\left(SX^{\sum_{i=1}^{m} \hat{R}_i}\right)^c = (SX^c)_{\sum_{i=1}^{m} \hat{R}_i}$;
4. $\mathrm{Bn}^{S}_{\sum_{i=1}^{m} \hat{R}_i}(X) = \mathrm{Bn}^{S}_{\sum_{i=1}^{m} \hat{R}_i}(X^c)$.

Proof The proofs of these terms are similar to Proposition 3.5. □

Proposition 3.8 *Let U be a universe and $\hat{R}_i (i = 1, ..., m)$ be m coverings of U induced by tolerance relations R_i $(i = 1, ..., m)$ and $X, Y \subseteq U$. Then the following properties hold:.*

1. $S(X \cap Y)_{\sum_{i=1}^{m} \hat{R}_i} = SX_{\sum_{i=1}^{m} \hat{R}_i} \cap SY_{\sum_{i=1}^{m} \hat{R}_i}$;
2. $S(X \cup Y)^{\sum_{i=1}^{m} \hat{R}_i} = SX^{\sum_{i=1}^{m} \hat{R}_i} \cup SY^{\sum_{i=1}^{m} \hat{R}_i}$;
3. $X \subseteq Y \Longrightarrow SX^{\sum_{i=1}^{m} \hat{R}_i} \subseteq SY^{\sum_{i=1}^{m} \hat{R}_i}$;
4. $X \subseteq Y \Longrightarrow SX_{\sum_{i=1}^{m} \hat{R}_i} \subseteq SY_{\sum_{i=1}^{m} \hat{R}_i}$;
5. $S(X \cap Y)^{\sum_{i=1}^{m} \hat{R}_i} \subseteq SX^{\sum_{i=1}^{m} \hat{R}_i} \cap SY^{\sum_{i=1}^{m} \hat{R}_i}$;
6. $S(X \cup Y)_{\sum_{i=1}^{m} \hat{R}_i} \supseteq SX_{\sum_{i=1}^{m} \hat{R}_i} \cup SY_{\sum_{i=1}^{m} \hat{R}_i}$.

Proof The proofs of these items are similar to Proposition 3.6. □

4 Difference and relationships among SGTRT, the 1st MGTRS, the 2nd MGTRS

From the above sections, we have known the concepts and properties of the 1st MGTRS and the 2nd MGTRS. We will investigate the difference and relationship among SGTRS, the 1st MGTRS, the 2nd MGTRS in this section.

Proposition 4.1 *Let U be a universe, $\hat{R}$ and $\hat{S}$ be two coverings of U induced by tolerance relations R and S and $X \subseteq U$. Then the following properties are true:*

1. $X_{\hat{R}} \cup X_{\hat{S}} = FX_{\hat{R}+\hat{S}}$;
2. $X^{\hat{R}} \cap X^{\hat{S}} = FX^{\hat{R}+\hat{S}}$;
3. $FX_{\hat{R}+\hat{S}} \subseteq X_{\hat{R}\cup\hat{S}} \subseteq X \subseteq X^{\hat{R}\cup\hat{S}} \subseteq FX^{\hat{R}+\hat{S}}$.

Proof

1. For any $x \in X_{\hat{R}} \cup X_{\hat{S}}$, we have

$$\begin{aligned} x \in X_{\hat{R}} \text{ or } x \in X_{\hat{S}} &\Longleftrightarrow \hat{R}(x) \subseteq X \text{ or } \hat{S}(x) \subseteq X \\ &\Longleftrightarrow x \in \{x : \hat{R}(x) \subseteq X \text{ or } \hat{S}(x) \subseteq X\} \\ &\Longleftrightarrow x \in FX_{\hat{R}+\hat{S}}, \end{aligned}$$

Hence, $X_{\hat{R}} \cup X_{\hat{S}} = FX_{\hat{R}+\hat{S}}$.

2. For any $x \in X^{\hat{R}} \cap X^{\hat{S}}$, we have

$$\begin{aligned} x \in X^{\hat{R}} \text{ and } x \in X^{\hat{S}} &\Longleftrightarrow \hat{R}(x) \cap X \neq \emptyset \text{ and } \hat{S}(x) \cap X \neq \emptyset \\ &\Longleftrightarrow x \in \{x : \hat{R}(x) \cap X \neq \emptyset \text{ and } \hat{S}(x) \cap X \neq \emptyset\} \\ &\Longleftrightarrow x \in FX^{\hat{R}+\hat{S}}, \end{aligned}$$

Hence, $X^{\hat{R}} \cap X^{\hat{S}} = FX^{\hat{R}+\hat{S}}$.

3. For any $x \in FX_{\hat{R}+\hat{S}}$, we know that $\hat{R}(x) \subseteq X$ or $\hat{S}(x) \subseteq X$. It is well known that $(\hat{R} \cup \hat{S})(x) \subseteq \hat{R}(x)$ and $(\hat{R} \cup \hat{S})(x) \subseteq \hat{S}(x)$, so $(\hat{R} \cup \hat{S})(x) \subseteq X$. Then we obtain $FX_{\hat{R}+\hat{S}} \subseteq X_{\hat{R}\cup\hat{S}}$.

□

On the other hand, for any $\in X^{\hat{R}\cup\hat{S}}$, $(\hat{R} \cup \hat{S})(x) \cap X \neq \emptyset$. So $\hat{R}(x) \cap X \neq \emptyset$ and $\hat{S}(x) \cap X \neq \emptyset$ hold. Then we can obtain $X^{\hat{R}\cup\hat{S}} \subseteq FX^{\hat{R}+\hat{S}}$.

It is obvious that $X_{\hat{R}\cup\hat{S}} \subseteq X \subseteq X^{\hat{R}\cup\hat{S}}$.

Therefore, (3) is proved.

Proposition 4.2 *Let U be a universe and $\hat{R}_i(i = 1, \ldots, m)$ be m coverings of U induced by tolerance relations R_i ($i = 1, \ldots, m$). Then we have*

1. $\bigcup_{i=1}^{m} X_{\hat{R}} = FX_{\sum_{i=1}^{m} \hat{R}_i}$;
2. $\bigcup_{i=1}^{m} X^{\hat{R}} = FX^{\sum_{i=1}^{m} \hat{R}_i}$;
3. $FX_{\sum_{i=1}^{m} \hat{R}_i} \subseteq X_{\bigcup_{i=1}^{m} \hat{R}_i} \subseteq X \subseteq X^{\bigcup_{i=1}^{m} \hat{R}_i} \subseteq FX^{\sum_{i=1}^{m} \hat{R}_i}$.

Proof The proof is similar to Proposition 4.1. □

Proposition 4.3 *Let U be a universe, $\hat{R}$ and $\hat{S}$ be two coverings of U induced by tolerance relations R and S and $X \subseteq U$. Then the following properties are true:.*

1. $X_{\hat{R}} \cap X_{\hat{S}} = SX_{\hat{R}+\hat{S}}$;
2. $X^{\hat{R}} \cup X^{\hat{S}} = SX^{\hat{R}+\hat{S}}$;
3. $SX_{\hat{R}+\hat{S}} \subseteq X_{\hat{R}\cup\hat{S}} \subseteq X \subseteq X^{\hat{R}\cup\hat{S}} \subseteq SX^{\hat{R}+\hat{S}}$.

Proof

1. For any $x \in X_{\hat{R}} \cap X_{\hat{S}}$, we have

$$\begin{aligned} x \in X_{\hat{R}} \text{ and } x \in X_{\hat{S}} &\Longleftrightarrow \hat{R}(x) \subseteq X \text{ and } \hat{S}(x) \subseteq X \\ &\Longleftrightarrow x \in \{x : \hat{R}(x) \subseteq X \text{ and } \hat{S}(x) \subseteq X\} \\ &\Longleftrightarrow x \in SX_{\hat{R}+\hat{S}}, \end{aligned}$$

Hence, $X_{\hat{R}} \cap X_{\hat{S}} = SX_{\hat{R}+\hat{S}}$.

2. For any $x \in X^{\hat{R}} \cup X^{\hat{S}}$, we have

$$\begin{aligned} x \in X^{\hat{R}} \text{ or } x \in X^{\hat{S}} &\Longleftrightarrow \hat{R}(x) \cap X \neq \emptyset \text{ or } \hat{S}(x) \cap X \neq \emptyset \\ &\Longleftrightarrow x \in \{x : \hat{R}(x) \cap X \neq \emptyset \text{ and } \hat{S}(x) \cap X \neq \emptyset\} \\ &\Longleftrightarrow x \in SX^{\hat{R}+\hat{S}}, \end{aligned}$$

Hence, $X^{\hat{R}} \cup X^{\hat{S}} = SX^{\hat{R}+\hat{S}}$.

3. For any $x \in SX_{\hat{R}+\hat{S}}$, we know that $\hat{R}(x) \subseteq X$ and $\hat{S}(x) \subseteq X$. It is well known that $(\hat{R} \cup \hat{S})(x) \subseteq \hat{R}(x)$ and $(\hat{R} \cup \hat{S})(x) \subseteq \hat{S}(x)$, so $(\hat{R} \cup \hat{S})(x) \subseteq X$. Then we obtain $SX_{\hat{R}+\hat{S}} \subseteq X_{\hat{R}\cup\hat{S}}$.

On the other hand, for any $\in X^{\hat{R}\cup\hat{S}}$, $(\hat{R} \cup \hat{S})(x) \cap X \neq \emptyset$. So $\hat{R}(x) \cap X \neq \emptyset$ and $\hat{S}(x) \cap X \neq \emptyset$ hold. Then we can obtain $X^{\hat{R}\cup\hat{S}} \subseteq SX^{\hat{R}+\hat{S}}$. □

It is obvious that $X_{\hat{R}\cup\hat{S}} \subseteq X \subseteq X^{\hat{R}\cup\hat{S}}$.

Therefore, $SX_{\hat{R}+\hat{S}} \subseteq X_{\hat{R}\cup\hat{S}} \subseteq X \subseteq X^{\hat{R}\cup\hat{S}} \subseteq SX^{\hat{R}+\hat{S}}$ holds.

Proposition 4.4 Let U be a universe, $\hat{R}_i(i = 1, \ldots, m)$ be m coverings of U induced by tolerance relations R_i ($i = 1, \ldots, m$) and $X \subseteq U$. Then we have

1. $\bigcap_{i=1}^{m} X_{\hat{R}} = SX_{\sum_{i=1}^{m} \hat{R}_i}$;
2. $\bigcup_{i=1}^{m} X_{\hat{R}} = SX^{\sum_{i=1}^{m} \hat{R}_i}$;
3. $SX_{\sum_{i=1}^{m} \hat{R}_i} \subseteq X_{\bigcap_{i=1}^{m}} \hat{R}_i \subseteq X \subseteq X_{\bigcap_{i=1}^{m}} \hat{R}_i \subseteq SX^{\sum_{i=1}^{m} \hat{R}_i}$.

Proof The proof is similar to Proposition 4.3. □

Proposition 4.5 Let U be a universe, $\hat{R}$ and $\hat{S}$ be two coverings of U induced by tolerance relations R and S and $X \subseteq U$. Then we have

1. $SX_{\hat{R}+\hat{S}} \subseteq FX_{\hat{R}+\hat{S}} \subseteq X_{\hat{R}\cap\hat{S}}$;
2. $SX^{\hat{R}+\hat{S}} \supseteq FX^{\hat{R}+\hat{S}} \supseteq X^{\hat{R}\cap\hat{S}}$.

Proof The proof can be obtained easily by Definitions 3.1, 3.3 and Propositions 4.1, 4.3. □

Proposition 4.6 Let U be a universe, $\hat{R}_i(i = 1, \ldots, m)$ be m coverings of U induced by tolerance relations R_i ($i = 1, \ldots, m$) and $X \subseteq U$. Then we have

1. $SX_{\sum_{i=1}^{m} \hat{R}_i} \subseteq FX_{\sum_{i=1}^{m} \hat{R}_i} \subseteq X_{\bigcap_{i=1}^{m}} X_{\hat{R}_i}$;
2. $SX^{\sum_{i=1}^{m} \hat{R}_i} \supseteq FX^{\sum_{i=1}^{m} \hat{R}_i} \supseteq X_{\bigcap_{i=1}^{m}} X_{\hat{R}_i}$.

Proof The proof can be obtained easily by Definitions 3.2, 3.4 and Propositions 4.2, 4.4. □

Proposition 4.7 Let U be a universe, $\hat{R}$ and $\hat{S}$ be two coverings of U induced by tolerance relations R and S and $X \subseteq U$. Then we have

1. $SX_{\hat{R}+\hat{S}} \subseteq X_{\hat{R}}(\text{ or } X_{\hat{S}}) \subseteq FX_{\hat{R}+\hat{S}}$;
2. $SX^{\hat{R}+\hat{S}} \supseteq X^{\hat{R}}(\text{ or } X^{\hat{S}}) \supseteq FX^{\hat{R}+\hat{S}}$.

Proof The proof can be obtained easily by Proposition 4.1(1), 4.3(1). □

Proposition 4.8 Let U be a universe, $\hat{R}_i(i = 1, \ldots, m)$ be m coverings of U induced by tolerance relations R_i (i = 1, ..., m) and $X \subseteq U$. Then we have

1. $SX_{\sum_{i=1}^{m}\hat{R}_i} \subseteq X_{\hat{R}_i} \subseteq FX_{\sum_{i=1}^{m}\hat{R}_i}$;
2. $SX^{\sum_{i=1}^{m}\hat{R}_i} \supseteq X^{\hat{R}_i} \supseteq FX^{\sum_{i=1}^{m}\hat{R}_i}$.

Proof The proof can be obtained easily by Propositions 4.2(1), 4.4(1). □

These propositions discuss about the relationships among the two types of MGTRS and SGTRS. Especially, from Propositions 4.7 and 4.8, the 2nd multi-granulation tolerance lower approximation is included in the single lower one and the first multi-granulation tolerance lower approximation includes the single lower one. For the upper approximations, the inclusion order is just in reverse. This properties reveals that the first MGTRS is coarser than the single one while the second MGTRS is more accurate. The following part about the measures of the 1st MGTRS and the 2nd MGTRS reflect this point further from the point of view of quantity.

5 Several measures in the first MGTRS and the second MGTRS

In the following, we will investigate several elementary measures in the first MGTRS and their properties.

Uncertainty of a set is due to the existence of a borderline region. The greater the borderline region of a set, the lower is the accuracy of the set (and vice versa). In order to express this idea more precisely, we introduce another accuracy measure as follows.

Definition 5.1 Let U be a universe, $\hat{R}_i(i = 1, \ldots, m)$ be m coverings of U induced by tolerance relations R_i (i = 1, ..., m) and $X \subseteq U$. The first rough measure of X by $\sum_{i=1}^{m}\hat{R}_i$ is defined as

$$\rho^F_{\sum_{i=1}^{m}\hat{R}_i}(X) = 1 - \frac{\left|FX_{\sum_{i=1}^{m}\hat{R}_i}\right|}{\left|FX^{\sum_{i=1}^{m}\hat{R}_i}\right|},$$

where $X \neq \emptyset$.

From the definitions, one can derive the following properties:

Proposition 5.1 Let U be a universe, $\hat{R}_i(i = 1, \ldots, m)$ be m coverings of U induced by tolerance relations R_i (i = 1, ..., m) and $X \subseteq U$. The following inequality

$$\rho_{\hat{R}_i}(X) \geq \rho^F_{\sum_{i=1}^{m}\hat{R}_i}(X) \geq \rho_{\bigcup_{i=1}^{m} X_{\hat{R}_i}}(X)$$

holds.

Proof Since $X_{\hat{R}_i} \subseteq FX_{\sum_{i=1}^{m}\hat{R}_i} \subseteq X_{\cap_{i=1}^{m}}X_{\hat{R}}$ and $X_{\cap_{i=1}^{m}}X_{\hat{R}_i} \subseteq FX^{\sum_{i=1}^{m}\hat{R}_i} \subseteq X^{\hat{R}_i}$, we can obtain that

$$\frac{\left|X_{\hat{R}_i}\right|}{\left|X^{\hat{R}_i}\right|} \leq \frac{\left|FX_{\sum_{i=1}^{m}\hat{R}_i}\right|}{\left|FX^{\sum_{i=1}^{m}\hat{R}_i}\right|} \leq \frac{\left|X_{\bigcup_{i=1}^{m} X_{\hat{R}_i}}\right|}{\left|X^{\bigcup_{i=1}^{m} X_{\hat{R}_i}}\right|};$$

then $\rho_{\hat{R}_i}(X) \geq \rho^F_{\sum_{i=1}^{m}\hat{R}_i}(X) \geq \rho_{\cup_{i=1}^{m} X_{\hat{R}_i}}(X)$ holds. □

Example 5.1 (*Continued from Example 3.1*) Computing the 1st rough measure of $X = \{3, 4, 5, 6\}$ by using the results in Example 3.1, it follows that

$$\rho_{\hat{R}_1}(X) = 1 - \frac{\left|X_{\hat{R}_1}\right|}{\left|X^{\hat{R}_1}\right|} = \frac{5}{6},$$

$$\rho_{\hat{R}_2}(X) = 1 - \frac{\left|X_{\hat{R}_2}\right|}{\left|X^{\hat{R}_2}\right|} = \frac{3}{5},$$

$$\rho_{\hat{R}_1\cup\hat{R}_2}(X) = 1 - \frac{\left|X_{\hat{R}_1\cup\hat{R}_2}\right|}{\left|X^{\hat{R}_1\cup\hat{R}_2}\right|} = 0,$$

$$\rho^F_{\hat{R}_1+\hat{R}_2}(X) = 1 - \frac{\left|FX_{\hat{R}_1+\hat{R}_2}\right|}{\left|FX^{\hat{R}_1+\hat{R}_2}\right|} = \frac{2}{5}.$$

Clearly, it follows from the earlier computation that

$$\rho_{\hat{R}_1}(X) \geq \rho^F_{\hat{R}_1+\hat{R}_2}(X) \geq \rho_{\hat{R}_1\cup\hat{R}_2}(X),$$

and

$$\rho_{\hat{R}_2}(X) \geq \rho^F_{\hat{R}_1+\hat{R}_2}(X) \geq \rho_{\hat{R}_1\cup\hat{R}_2}(X).$$

Definition 5.2 Let U be a universe, $\hat{S}$ be the covering induced by tolerance relation S, and $\hat{R} = \{\hat{R}_i, i = 1, \ldots, m\}$ be m coverings induced by tolerance relations R_i, i = 1, ..., m. The quality of approximation of $\hat{S}$ by $\hat{R}$, also called the first degree of dependency, is defined by

$$\gamma_F\left(\sum_{i=1}^{m}\hat{R}_i, \hat{S}\right) = \frac{\Sigma\left\{\left|FX_{\sum_{i=1}^{m}\hat{R}_i}\right| : X \in \hat{S}\right\}}{|U|},$$

and is used to evaluate the deterministic part of the tolerance rough set description of $\hat{S}$ by counting those elements which can be reclassified into blocks of $\hat{S}$ which is the knowledge given by $\sum_{i=1}^{m}\hat{R}_i$.

Proposition 5.2 Let U be a universe, $\hat{S}$ be the covering induced by tolerance relation S, and $\hat{R} = \{\hat{R}_i, i = 1, ..., m\}$ be m coverings induced by tolerance relations R_i, $i = 1, ..., m$. The inequalities

$$\gamma(\hat{R}_i, \hat{S}) \leq \gamma_F\left(\sum_{i=1}^{m} \hat{R}_i, \hat{S}\right) \leq \gamma\left(\bigcup_{i=1}^{m} X_{\hat{R}_i}, \hat{S}\right)$$

are true.

Proof It can be proved by using a similar method as Proposition 5.1. □

In the following, we will investigate several elementary measures in the second MGTRS and their properties.

Similarly, we introduce the accuracy measure to the second MGTRS as follows:

Definition 5.3 Let U be a universe, $\hat{R}_i (i = 1, ..., m)$ be m coverings of U induced by tolerance relations R_i ($i = 1, ..., m$) and $X \subseteq U$. The second rough measure of X by $\sum_{i=1}^{m} \hat{R}_i$ is defined as

$$\rho^S_{\sum_{i=1}^{m} \hat{R}_i}(X) = 1 - \frac{\left|SX_{\sum_{i=1}^{m} \hat{R}_i}\right|}{\left|SX^{\sum_{i=1}^{m} \hat{R}_i}\right|},$$

where $X \neq \emptyset$.

From the definitions, one can derive the following properties:

Proposition 5.3 Let U be a universe, $\hat{R}_i (i = 1, ..., m)$ be m coverings of U induced by tolerance relations R_i ($i = 1, ..., m$) and $X \subseteq U$. Then we have

$$\rho^S_{\sum_{i=1}^{m} \hat{R}_i}(X) \geq \rho_{\hat{R}_i}(X) \geq \rho_{\cup_{i=1}^{m} X_{\hat{R}_i}}(X).$$

Proof Since $SX_{\sum_{i=1}^{m} \hat{R}_i} \subseteq X_{\hat{R}_i} \subseteq X_{\cap_{i=1}^{m}} X_{\hat{R}_i}$ and $X^{\cap_{i=1}^{m}} X_{\hat{R}_i} \subseteq X^{\hat{R}_i} \subseteq SX^{\sum_{i=1}^{m} \hat{R}_i}$ by Definition 3.4 and Proposition 4.4, we can obtain that

$$\frac{\left|SX_{\sum_{i=1}^{m} \hat{R}_i}\right|}{\left|SX^{\sum_{i=1}^{m} \hat{R}_i}\right|} \leq \frac{\left|X_{\hat{R}_i}\right|}{\left|X^{\hat{R}_i}\right|} \leq \frac{\left|X_{\cup_{i=1}^{m} X_{\hat{R}_i}}\right|}{\left|X^{\cup_{i=1} m X_{\hat{R}_i}}\right|};$$

then $\rho^S_{\sum_{i=1}^{m} \hat{R}_i}(X) \geq \rho_{\hat{R}_i}(X) \geq \rho_{X_{\cap_{i=1}^{m}}} X_{\hat{R}_i}(X)$ holds. □

Proposition 5.4 Let U be a universe, $\hat{R}_i (i = 1, ..., m)$ be m coverings of U induced by tolerance relations R_i ($i = 1, ..., m$) and $X \subseteq U$. Then we have

$$\rho^S_{\sum_{i=1}^{m} \hat{R}_i}(X) \geq \rho_{\hat{R}_i}(X) \geq \rho^F_{\sum_{i=1}^{m} \hat{R}_i}(X).$$

Proof It can be obtained directly by Propositions 5.1 and 5.3. □

Example 5.2 (*Continued from Example 3.2*) Computing the 2nd rough measure of $X = \{3, 4, 5, 6\}$ by using the results in Example 3.2, it follows that

$$\rho_{\hat{R}_1}(X) = 1 - \frac{\left|X_{\hat{R}_1}\right|}{\left|X^{\hat{R}_1}\right|} = \frac{5}{6},$$

$$\rho_{\hat{R}_2}(X) = 1 - \frac{\left|X_{\hat{R}_2}\right|}{\left|X^{\hat{R}_2}\right|} = \frac{3}{5},$$

$$\rho_{\hat{R}_1 \cup \hat{R}_2}(X) = 1 - \frac{\left|X_{\hat{R}_1 \cup \hat{R}_2}\right|}{\left|X^{\hat{R}_1 \cup \hat{R}_2}\right|} = 0,$$

$$\rho^S_{\hat{R}_1 + \hat{R}_2}(X) = 1 - \frac{\left|SX_{\hat{R}_1 + \hat{R}_2}\right|}{\left|SX^{\hat{R}_1 + \hat{R}_2}\right|} = 1.$$

Clearly, it follows from the earlier computation that

$$\rho^S_{\hat{R}_1 + \hat{R}_2}(X) \geq \rho_{\hat{R}_1}(X) \geq \rho_{\hat{R}_1 \cup \hat{R}_2}(X),$$

and

$$\rho^S_{\hat{R}_1 + \hat{R}_2}(X) \geq \rho_{\hat{R}_2}(X) \geq \rho_{\hat{R}_1 \cup \hat{R}_2}(X).$$

Definition 5.4 Let U be a universe, $\hat{S}$ be the covering induced by tolerance relation S, and $\hat{R} = \{\hat{R}_i, i = 1, ..., m\}$ be m coverings induced by tolerance relations R_i, $i = 1, ..., m$. The quality of approximation of $\hat{S}$ by $\hat{R}$, also called the second degree of dependency, is defined by

$$\gamma_S\left(\sum_{i=1}^{m} \hat{R}_i, \hat{S}\right) = \frac{\Sigma\left\{\left|SX_{\sum_{i=1}^{m} \hat{R}_i}\right| : X \in \hat{S}\right\}}{|U|},$$

and is used to evaluate the deterministic part of the tolerance rough set description of $\hat{S}$ by counting those elements which can be reclassified to blocks of $\hat{S}$ which is the knowledge given by $\sum_{i=1}^{m} \hat{R}_i$.

Proposition 5.5 Let U be a universe, $\hat{S}$ be the covering induced by tolerance relation S, and $\hat{R} = \{\hat{R}_i, i = 1, ..., m\}$ be m coverings induced by tolerance relations R_i, $i = 1, ..., m$. The inequalities

$$\gamma_S\left(\sum_{i=1}^{m} \hat{R}_i, \hat{S}\right) \leq \gamma(\hat{R}_i, \hat{S}) \leq \gamma\left(\bigcup_{i=1}^{m} X_{\hat{R}_i}, \hat{S}\right)$$

are true.

Proof It can be proved by using a similar method as Proposition 5.3. □

Proposition 5.6 Let U be a universe, $\hat{S}$ be the covering induced by tolerance relation S, and $\hat{R} = \{\hat{R}_i, i = 1, ..., m\}$ be m coverings induced by tolerance relations R_i, $i = 1, ..., m$. The inequalities as follows holds:

$$\gamma_S\left(\sum_{i=1}^{m} \hat{R}_i, \hat{S}\right) \leq \gamma(\hat{R}_i, \hat{S}) \leq \gamma_F\left(\sum_{i=1}^{m} \hat{R}_i, \hat{S}\right)$$

Proof It can be obtained directly by Propositions 5.2 and 5.5. □

6 Conclusion

On the basis of classical rough set theory which defined the lower and upper approximations by using an equivalence relation, some researchers proposed its extended model, called tolerance rough set model, using a tolerance relation. Nevertheless, by relaxing the indiscernibility relation to more general binary relations, more improved rough set models have been successfully applied for knowledge representation. The contribution of this correspondence paper is to extend the tolerance rough set model to two types of tolerance multi-granulation rough set models which are based on multiple tolerance relations. In this paper, two types of Multi-granulation rough set models have been constructed. In particular, some important properties of the two types of Multi-granulation rough set models are investigated and the difference and relationship among tolerance single-granulation rough set, first MGTRS and second MGTRS are shown. Several examples are given to illustrate the two types of tolerance rough set models. Moreover, several important measures have been developed in two types of MGTRS. Due to the definitions and corresponding properties of the two types of MGTRS, we can deal with some problems in information systems possessing contradiction or inconsistent relationships. The two types of tolerance Multi-granulation rough set models are useful for solving many complex practical applications.

Acknowledgments This paper is supported by National Natural Science Foundation of China (No.61105041,71071124 and 1100 1227), Natural Science Foundation Project of CQ CSTC (No.cstc 2011jjA40037), and Science and Technology Program of Board of Education of Chongqing (KJ120805).

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